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DISPERSIVE BEHAVIOURS IN COMPLEX STRUCTURES

BY

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Abstract. In the framework of the Scale Relativity Theory, a fractal Korteweg-de Vries-type equation for describing dispersive behaviours of any complex structure is obtained. In the one-dimensional case, the general solution of this equation show that the dispersive behaviours of any complex structure are given by means of cnoidal oscillation modes. Through these modes degeneration, dispersive behaviours of soliton, soliton-package, harmonic etc. result.

Keywords: dispersive behaviours; Scale Relativity Theory; fractal Korteweg-de Vries-type equation; complex structures.

1. Introduction

Complex structures are structures in which the collective behavior of their components involves the emergence of properties that can be hardly (or not at all) deducted from the individual properties of said components. Perhaps the most famous quote for what would be defined as a „complex system” (structure) belongs to Aristotle: „The whole is more than the sum of its parts”. Therefore, this collective behavior cannot be analyzed or predicted only through the (complete) knowledge of the properties of individual components (Mitchell, 2009).

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Complex structures usually consist of many components, which interact in various ways with each other and possibly with their environment also. These components could form interactions networks. The different types of interactions could give rise to new information, making it difficult to separately study components or to exactly predict their evolution. Moreover, the components of such a system can also be viewed as entirely new systems, this leading to systems of systems that are interdependent on each other. Thus, one of the core challenges of complexity science is not only to analyze the parts and their interactions but also to fully understand how connections form and generate a whole (Bar-Yam, 1997).

In relatively simple structures, the properties of the entire structure can be understood from its components addition or aggregation. Therefore, a simple structure's macroscopic properties can be deduced from the properties of its separate parts at the microscopic scale. In complex structures, however, we can often be in a situation in which the properties of the whole cannot be understood or predicted solely based on the knowledge of its components, due to a phenomenon called "emergence" (Ball, 2004).

We must also note that structures (systems) can also be analyzed by tracking the changes of their states over time. A state can be described in variables sets that characterize a given system. As it changes states, its variables also change, as a "reaction" to its environment. We say that the change is linear if it is directly proportional to time, the actual state of the system, or environmental changes, or that it is non-linear if it is not proportional to them. Complex systems (structures) are usually non-linear, changing at different rates with respect to their states and environment. They could also exhibit stable states at which they can stay the same even if perturbations manifest, or unstable states in which the systems can be disrupted by a small perturbation. In some rare cases, even small environmental changes can seriously alter a system's behavior. Some systems are chaotic, meaning that they are very sensitive to small perturbations and their behavior can be unpredictable, displaying a "butterfly effect" (Gleick, 2011).

Interactions among complex structures' components can give rise to global patterns or behaviors. This can be described as self-organization, because no central/external controller exists. Self-organization can generate physical/functional structures such as crystalline patterns of materials and various morphologies of biological organisms, or dynamic/informational behaviors such as the fish's shoaling behaviors and electrical impulses in animal muscles. Sometimes, complex structures can self-organize into a "critical" state, which can exist only in a fragile balance between its stochastic and deterministic behavior (Ball, 1999).

Complex structures are usually active and they respond to the environment. This adaptation can take place at multiple scales: cognitive, through the development of learning and psychological functions; social, like information sharing through social interactions and dynamics; or even evolutionary, through genetics and natural selection. In cases when the components suffer damage or are removed, such structures are able to adapt and recover their previous functionalities, sometimes becoming better than before. Complex structures that display such properties are called complex adaptive structures (Holland, 1992).

Complex structures can be found in various domains, such as physics, biology, social sciences, finance, politics, psychology, medicine, engineering, information technology etc. Many state-of-the-art technologies, applied to social media, mobile communications, autonomous vehicles, or blockchains, generate complex structures with emergent properties. A fundamental concept of complexity science is the idea that different systems in various domains display phenomena with common features that can be described using the same scientific models – this is called universality. These concepts highlight the need for a new multidisciplinary mathematical and computational framework. Therefore, complexity science can provide a comprehensive and multi-disciplinary analytical approach, thus complementing traditional scientific methods that focus on specific subjects in each domain (Thurner *et al.*, 2018).

In this paper we analyze some dispersive behaviors in complex structures through fractal Korteweg-de Vries-type equations.

2. Mathematical Model

Let us write the equation of motion in its covariant form (geodesic equations)

$$\frac{\hat{d}\hat{\boldsymbol{V}}}{dt}=\frac{\partial\hat{\boldsymbol{V}}}{\partial t}+\left(\hat{\boldsymbol{V}}\cdot\nabla\right)\hat{\boldsymbol{V}}-i\lambda\left(dt\right)^{(2/D_F)-1}\Delta\hat{\boldsymbol{V}}+ +\frac{\sqrt{2}}{3}\lambda^{3/2}\left(dt\right)^{(3/D_F)-1}\nabla^3\hat{\boldsymbol{V}}=0 \tag{1}$$

In Eq. (1) we can see that, in any point on a fractal path, the local acceleration, $\partial_t\hat{\boldsymbol{V}}$, convection, $\left(\hat{\boldsymbol{V}}\nabla\right)\hat{\boldsymbol{V}}$, dissipation, $\lambda\left(dt\right)^{(2/D_F)-1}\Delta\hat{\boldsymbol{V}}$, and dispersion, $\lambda^{3/2}\left(dt\right)^{(3/D_F)-1}\nabla^3\hat{\boldsymbol{V}}$ are in equilibrium. The dissipative and dispersive terms in (1) specify that the behaviors of the complex structure are of a viscoelastic or hysteretic type.

Due to the fact that interactions in the complex structure are not present, we can practically employ self-convection, self-dissipation and self-dispersion-type mechanisms (i.e., we can describe the dynamics of any complex structure by means of a fractal-type fluid). Thus, the geodesics equations can be identified with the complex structure's streamlines.

Using a standard method on Eq. (1), we obtain, at the differentiable resolution scale:

$$\frac{\hat{d}\hat{\mathbf{V}}_D}{dt}=\frac{\partial \mathbf{V}_D}{\partial t}+\left(\mathbf{V}_D\cdot\nabla\right)\mathbf{V}_D-\left(\mathbf{V}_F\cdot\nabla\right)\mathbf{V}_F-\lambda\ \left(dt\right)^{\left(2/D_F\right)-1}\Delta\mathbf{V}_F+ \\ +\frac{\sqrt{2}}{3}\lambda^{3/2}\left(dt\right)^{\left(3/D_F\right)-1}\nabla^3\mathbf{V}_D=0 \tag{2}$$

and, also, at the fractal resolution scale:

$$\frac{\hat{d}\mathbf{V}_F}{dt}=\frac{\partial \mathbf{V}_F}{\partial t}+\left(\mathbf{V}_F\cdot\nabla\right)\mathbf{V}_D-\left(\mathbf{V}_D\cdot\nabla\right)\mathbf{V}_F- \\ -\lambda\left(dt\right)^{\left(2/D_F\right)-1}\Delta\mathbf{V}_D+\frac{\sqrt{2}}{3}\lambda^{3/2}\left(dt\right)^{\left(3/D_F\right)-1}\nabla^3\mathbf{V}_F=0 \tag{3}$$

For irotational motions:

$$\nabla\times\hat{\boldsymbol{V}}=0,\nabla\times\boldsymbol{V}_D=0,\nabla\times\boldsymbol{V}_F=0 \tag{4}$$

The velocity field can be written as:

$$\hat{\boldsymbol{V}}=\text{-}2i\lambda\left(dt\right)^{\left(2/D_F\right)-1}\nabla\ln\psi \tag{5}$$

or explicitly, with $\psi=\sqrt{\rho}\exp\left(iS\right)$,

$$\begin{gathered}\hat{\boldsymbol{V}}=2\lambda dt^{\left(2/D_F\right)-1}\nabla S-i2\lambda\left(dt\right)^{\left(2/D_F\right)-1}\nabla\ln\rho \\ \boldsymbol{V}_D=2\lambda\left(dt\right)^{\left(2/D_F\right)-1}\nabla S \\ \boldsymbol{V}_F=2\lambda\left(dt\right)^{\left(2/D_F\right)-1}\nabla\ln\rho\end{gathered} \tag{6}$$

where $\sqrt{\rho}$ is an amplitude and S a phase.

In this context, ρ=const., and thus Eqs. (2) and (3) take the single form:

$$\frac{\hat{d}\boldsymbol{V}_D}{dt}=\frac{\partial \boldsymbol{V}_D}{\partial t}+\left(\boldsymbol{V}_D\cdot\nabla\right)\boldsymbol{V}_D+\frac{\sqrt{2}}{3}\lambda^{3/2}\left(dt\right)^{\left(3/D_F\right)-1}\nabla^3\boldsymbol{V}_D=0 \tag{7}$$

Eq. (7) is a generalization of the standard Korteweg-de Vries equation for dynamics on fractal manifolds.

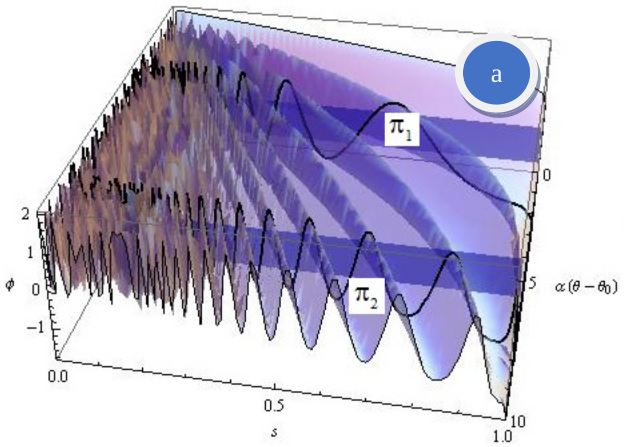
In order to obtain a solution for $\boldsymbol{V}_D$ in the one-dimensional case, it is necessary to introduce the dimensionless variables

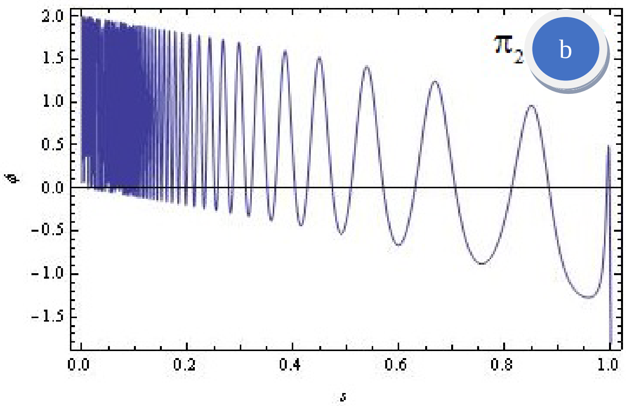
$$\omega t = \tau_1, kx = \xi_1, \theta = \xi_1 - M\tau_1, \frac{V_D}{c} = \Phi \tag{8}$$

Then, the solution of Eq. (7) becomes (for details see (Jackson, 1993; Merches and Agop, 2015; Whitham, 1974)):

$$\Phi = \bar{\Phi} + 2a\left[\frac{E(s)}{K(s)} - 1\right] + 2acn^2\left[\alpha(\theta - \theta_0); s\right] \tag{9}$$

It follows that a one-dimensional space-time dynamics for the complex structure can be obtained by means of the cnoidal oscillations modes of the normalized velocity field – Figs. 1a-c and 2a-f.





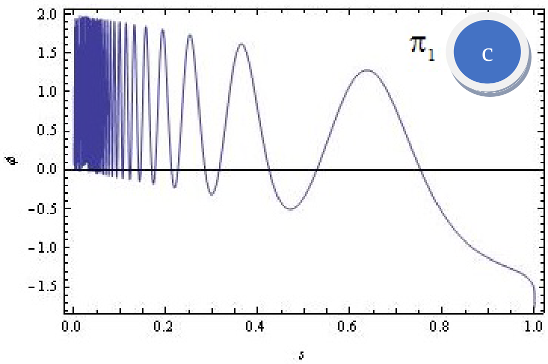


Fig. 1 – Three – dimensional (a) and two – dimensional (b, c) cnoidal oscillation modes of a velocity field.

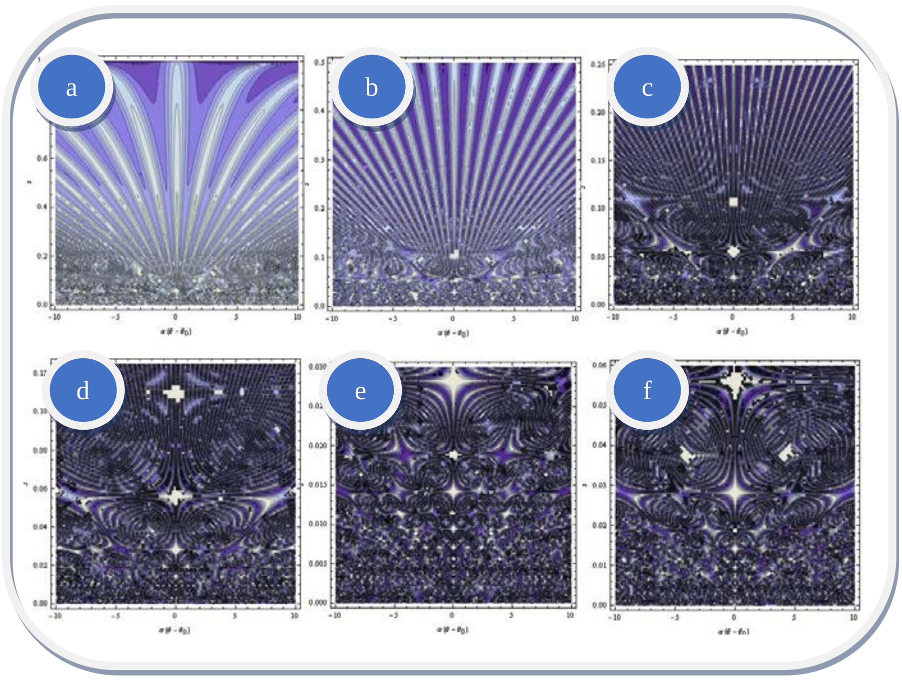


Fig. 2 – Fractal behaviors of the normalized velocity field by means of self-similarity. Contour plots for various non-linearity degrees.

The physical meanings of quantities from relations (8) and (9) are given in (Mercheș and Agop, 2015). In addition, let us note that *cn* is the Jacobi cnoidal elliptical function of modulus *s* (Armitage, 2006) and argument $\alpha\left(\theta-\theta_0\right)$ with $\theta_0=\text{const.}$

The cnoidal oscillation modes have the following characteristic parameters:

i) Wave number

$$k = \frac{\pi a^{1/2}}{sK(s)} \tag{10}$$

ii) Phase velocity

$$U = 6\bar{\Phi} + 4a\left[\frac{3E(s)}{K(s)} - \frac{1+s^2}{s^2}\right] \tag{11}$$

iii) Quasi-period (see Figs. 3a,b)

$$T = \frac{1}{\dfrac{3\bar{\Phi}a^{1/2}}{sK(s)} + \dfrac{2a^{3/2}}{sK(s)}\left[\dfrac{3E(s)}{K(s)} - \dfrac{1+s^2}{s^2}\right]} \tag{12}$$

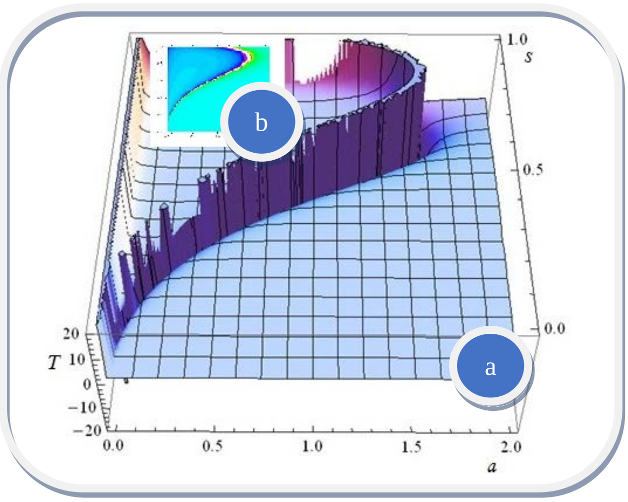


Fig. 3 – Quasi period of the cnoidal oscillation modes versus amplitude and non-linearity degree (a); two dimensional contour of the quasi-period (b).

Through degenerations of the cnoidal oscillation modes, it results:

For s→0, (9) reduces to harmonic package-type sequence (Fig. 4a)

$$\Phi \approx \bar{\Phi} + a + a\cos\left[k\alpha(\theta - \theta_0)\right] \tag{13}$$

characterized by wave number

$$k \approx \frac{2a^{1/2}}{s} \tag{14}$$

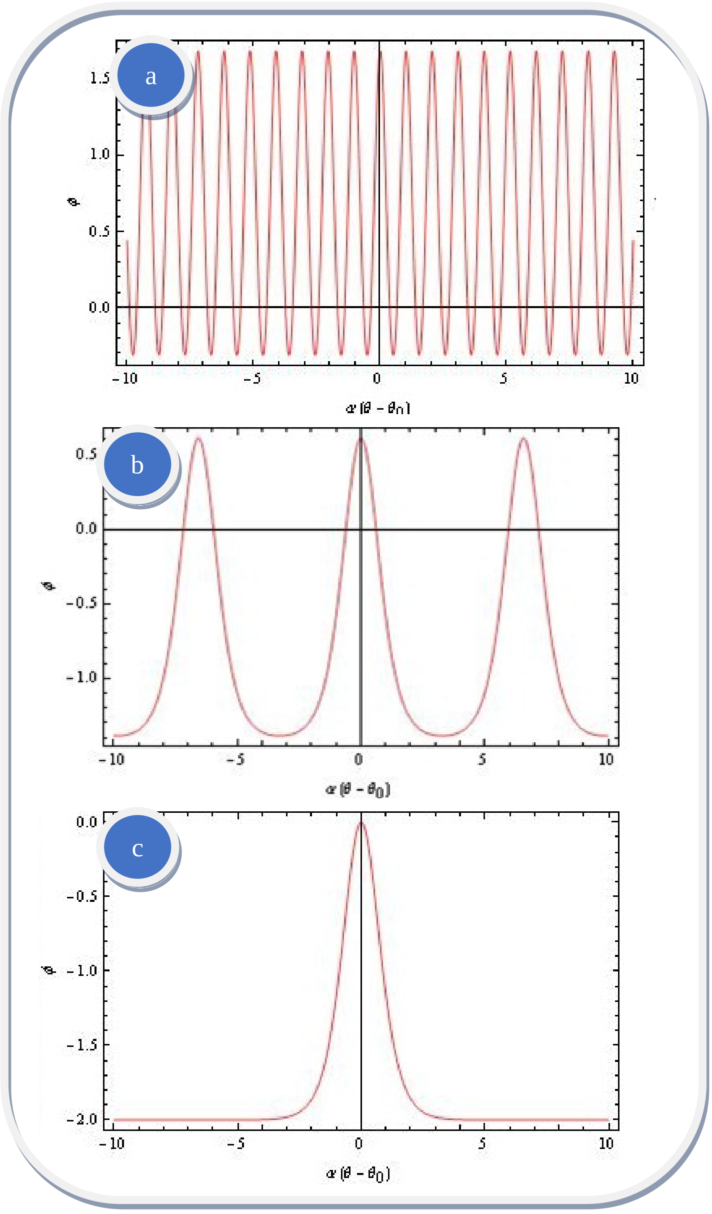


Fig. 4 – Pure sequences obtained through degenerations of cnoidal oscillations modes of velocity field: harmonic package – type sequence (a), soliton package – type sequence (b), soliton – type sequence (c).

phase velocity

$$U \approx 6\bar{\Phi} + 8a - k^2 \quad (15)$$

and pulsation

$$\Omega = 2\pi/T \approx 6\bar{\Phi}k + 8ak - k^3 \quad (16)$$

i) For s→1, (9) reduces to a soliton-package-type sequence (Fig. 4b)

$$\Phi \approx \bar{\Phi} + a_1 \operatorname{sech}^2\left[\left(\frac{a_1}{6}\right)^{1/2}\left(\theta - \theta_0\right)\right] \quad (17)$$

characterized by wave number

$$\Lambda \approx \frac{\left(2a_1\right)^{1/2}}{4k_1}, a_1 = 2a, k_1 = \frac{k}{2\pi} \quad (18)$$

phase velocity

$$U \approx 6\bar{\Phi} + 2a_1 - 12k_1\left(a_1\right)^{1/2} \quad (19)$$

and pulsation

$$\Omega \approx 12\pi\bar{\Phi}k_1 + 4\pi a_1 k_1 - 24\pi k_1^2\left(a_1\right)^{1/2} \quad (20)$$

For s = 0, (9) reduces to a harmonic type sequence, while for s = 1 to a soliton type one (Fig. 4c).

The degenerations of cnoidal oscillation modes contain the following sequential mixtures: harmonic type sequence – harmonic package type sequence, soliton type sequence–soliton package type sequence etc. Such situations can be found if we assume that non-linearity s depends on the resolution scale. Examples of such types of oscillations can be found in (Nica *et al.*, 2012; Pompilian *et al.*, 2013). In the case of a harmonic packet, Eq. (12) indicates a chirping type effect (Cristescu, 2008).

Eliminating the amplitude a, between (10) and (11) we can write:

$$\left(U - 6\bar{\Phi}\right)\lambda^2 = 16A(s),\ k = \frac{2\pi}{\lambda} \quad (21)$$

where

$$A(s) = 3s^2K(s)E(s) - \left(1 + s^2\right)K^2(s) \quad (22)$$

3. Results

By analyzing the degenerations of the cnoidal oscillation modes, we can see two distinct flow regimes of the dissipative complex fluid: non-quasi-autonomous regime and quasi-autonomous regime (Fig. 5). In Fig. 5, the value $s \approx 0.7$ separates the two flow regimes, as it results from the following equation:

$$\left(U-6\bar{\Phi}\right)\lambda^{2} \approx const \tag{23}$$

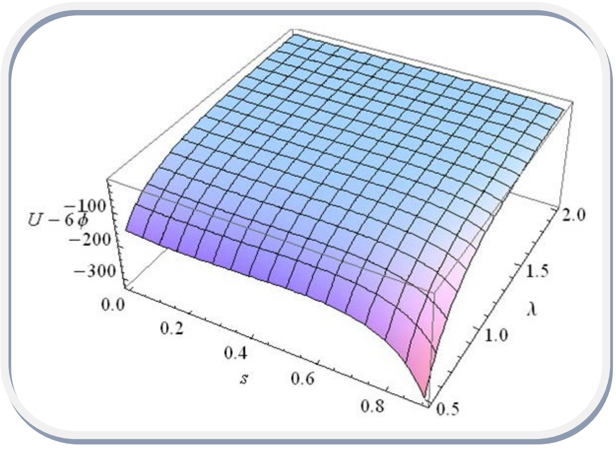


Fig. 5 – Flows regimes of the complex fluid for different non-linearity degrees.

We should note here that the one – dimensional space – time lattices of nonlinear oscillators can be associated to cnoidal oscillation modes, *i.e.* Toda lattices (Toda, 1981).

4. Conclusions

Assuming that the motion of complex structures' particles take place on continuous and non-differentiable curves, the geodesics equations in fractal space are obtained. In cases in which the dissipative effects are negligible compared to the convective and dispersive ones, its flow dynamics are given through space – time cnoidal oscillation modes of the complex velocity field.

Harmonic, harmonic packet, soliton, soliton packet sequences can therefore be obtained through space – time cnoidal oscillations modes degenerations.

However, we must stress the fact that nature does not operate with the afore-mentioned pure sequences, but with mixture sequences as harmonic – harmonic packet, soliton – soliton packet etc. The self-similarity of the cnoidal modes specifies the existence of some "cloning" mechanisms (full and fractional velocity function – a function which evolves in time to a state describable as a collection of spatially distributed sub-velocity-functions that each closely reproduces the initial velocity-function shape (Aronstein, 1997)). All these show a direct connection between the fractal structure of the flow dynamics of complex fluid and holographic behaviours (Butuc *et al.*, 2016).

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COMPORTAMENTE DISPERSIVE ÎN STRUCTURI COMPLEXE

(Rezumat)

În Teoria Relativității de Scală, se obține o ecuație fractală de tip Korteweg-de Vries, ecuație utilizată în descrierea unor comportamente dispersive ale structurilor complexe. În cazul unidimensional, soluția unei astfel de ecuații este dictată de moduri cnoidale de oscilație, prin degenerarea cărora rezultă comportamente de tip soliton, pachet de solitoni, armonic etc.